



---

**MURI-09) MULTI-SCALE FUSION OF INFORMATION FOR UNCERTAINTY QUANTIFICATION AND M**

**George Em Karniadakis**  
**BROWN UNIVERSITY IN PROVIDENCE IN STATE OF RI AND PROVIDENCE PLANTATIONS**

---

**12/02/2015**  
**Final Report**

**DISTRIBUTION A: Distribution approved for public release.**

**Air Force Research Laboratory**  
**AF Office Of Scientific Research (AFOSR)/ RTA2**  
**Arlington, Virginia 22203**  
**Air Force Materiel Command**

<b>REPORT DOCUMENTATION PAGE</b>		Form Approved OMB No. 0704-0188	
<p>The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Executive Services, Directorate (0704-0188). Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.</p> <p>PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ORGANIZATION.</p>			
<b>1. REPORT DATE (DD-MM-YYYY)</b> 03-12-2015		<b>2. REPORT TYPE</b> Final Performance	
		<b>3. DATES COVERED (From - To)</b> 01-09-2009 to 31-08-2015	
<b>4. TITLE AND SUBTITLE</b> MULTI-SCALE FUSION OF INFORMATION FOR UNCERTAINTY QUANTIFICATION AND MANAGEMENT IN LARGE-SCALE SIMULATIONS		<b>5a. CONTRACT NUMBER</b>	
		<b>5b. GRANT NUMBER</b> FA9550-09-1-0613	
		<b>5c. PROGRAM ELEMENT NUMBER</b> 61102F	
<b>6. AUTHOR(S)</b> George Em Karniadakis		<b>5d. PROJECT NUMBER</b>	
		<b>5e. TASK NUMBER</b>	
		<b>5f. WORK UNIT NUMBER</b>	
<b>7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)</b> BROWN UNIVERSITY IN PROVIDENCE IN STATE OF RI AND PROVIDENCE PLANTATIONS 1 PROSPECT STREET PROVIDENCE, RI 02912-9079 US		<b>8. PERFORMING ORGANIZATION REPORT NUMBER</b>	
<b>9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)</b> AF Office of Scientific Research 875 N. Randolph St. Room 3112 Arlington, VA 22203		<b>10. SPONSOR/MONITOR'S ACRONYM(S)</b> AFRL/AFOSR RTA2	
		<b>11. SPONSOR/MONITOR'S REPORT NUMBER(S)</b>	
<b>12. DISTRIBUTION/AVAILABILITY STATEMENT</b> A DISTRIBUTION UNLIMITED: PB Public Release			
<b>13. SUPPLEMENTARY NOTES</b>			
<b>14. ABSTRACT</b> <p>We developed an integrated methodology for uncertainty quantification (UQ) that proceeds from initial problem definition to engineering applications. We worked on five research areas: (1) Mathematical analysis of Stochastic Partial Differential Equations (SPDEs) and multiscale formulation; (2) Numerical solution of SPDEs; (3) Reduced-Order modeling; (4) Estimation/Inverse problems; and (5) Robust optimization and control. This work sets the mathematical foundations of Uncertainty Quantification methods used by many diverse communities in computational mechanics, fluid dynamics, plasma dynamics, and materials science. We have pioneered methods for efficient high-dimensional representations of stochastic processes, established Wick-Malliavin approximation for nonlinear SPDEs, theoretical error estimates for multiscale parametric and stochastic PDEs, a new approach to design of experiment and UQ on parametric manifolds, multi-fidelity optimization-under-uncertainty, a data-driven Bayesian framework and probabilistic graphical models for UQ, and information-based coarse graining methods</p>			
<b>15. SUBJECT TERMS</b> FUSION, SIMULATIONS			

Standard Form 298 (Rev. 8/98)  
Prescribed by ANSI Std. Z39.18

DISTRIBUTION A: Distribution approved for public release

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			George Em Kamiadakis
Unclassified	Unclassified	Unclassified	UU		19b. TELEPHONE NUMBER (Include area code) 401-863-1217

# MULTI-SCALE FUSION OF INFORMATION FOR UNCERTAINTY QUANTIFICATION AND MANAGEMENT IN LARGE-SCALE SIMULATIONS

## AFOSR GRANT NUMBER: FA9550-09-1-0613

### (FINAL REPORT)

GE Karniadakis, JS Hesthaven & B Rozovsky, *Brown University*; AT Patera & K Willcox, *MIT*; N Zabaras, *Cornell University*; T Hou, *Caltech*

#### Abstract

We developed an integrated methodology for uncertainty quantification (UQ) that proceeds from initial problem definition to engineering applications. Towards this goal, we worked on five research areas: (1) Mathematical analysis of SPDEs and multiscale formulation; (2) Numerical solution of SPDEs; (3) Reduced-Order modeling; (4) Estimation/Inverse problems; and (5) Robust optimization and control. This work set the mathematical foundations of Uncertainty Quantification methods used by many diverse communities in computational mechanics, fluid dynamics, plasma dynamics, and materials science. We have pioneered methods for efficient high-dimensional representations of stochastic processes, established Wick-Malliavin approximation for nonlinear SPDEs, theoretical error estimates for multiscale parametric and stochastic PDEs, a new approach to design of experiment and UQ on parametric manifolds, multi-fidelity optimization-under-uncertainty, a data-driven Bayesian framework and probabilistic graphical models for UQ, and information-based coarse graining methods. We have also demonstrated an integration of our UQ methodology and all five areas for a benchmark problem. We have published more than **150 papers** in top mathematical journals, obtained one patent (MIT), and have established one software company (MIT).

## Contents

<b>1</b>	<b>Status/Progress</b>	<b>2</b>
1.1	Mathematical analysis of SPDEs and multiscale formulation . . . . .	2
1.2	Numerical solution of SPDEs . . . . .	3
1.3	Reduced-Order modeling . . . . .	5
1.4	Estimation/Inverse problems . . . . .	6
1.5	Robust optimization and control . . . . .	7
1.6	Integrated UQ Methodology . . . . .	8
<b>2</b>	<b>Personnel Supported During Duration of Grant</b>	<b>9</b>
<b>3</b>	<b>Honors &amp; Awards</b>	<b>9</b>
<b>4</b>	<b>AFRL Point of Contact</b>	<b>9</b>
<b>5</b>	<b>Transitions</b>	<b>9</b>
<b>6</b>	<b>Acknowledgement/Disclaimer</b>	<b>10</b>
<b>7</b>	<b>Publications</b>	<b>10</b>

# 1 Status/Progress

In the following we provide some research highlights in each of the five research areas of the MURI.

## 1.1 Mathematical analysis of SPDEs and multiscale formulation

**Wick Malliavin Approximations to nonlinear Stochastic PDEs (Brown leads)** An important achievement is the development of completely new nonlinear Malliavin calculus. This type of calculus is important for the analysis and simulation of stationary and/or “causal” systems. It allows effective treatment of systems perturbed by nonlinear functions of colored Gaussian noise. We have also developed an effective methodology for homogenization of random elliptic PDEs with deterministic multi-scale structure of the coefficients.

Approximating nonlinearities in Stochastic Partial Differential Equations (SPDEs) via the Wick product has often been used in quantum field theory and stochastic analysis. The main benefit is simplification of the equations but at the expense of introducing modeling errors. We have shown that the Wick solutions have accuracy comparable to linear stochastic perturbation series solutions. However, number-theoretical renormalizations, e.g., based on Catalan numbers for the Wick-Navier-Stokes (WNS) equations, can improve the accuracy by orders of magnitude. The propagator of Wick approximations to nonlinear SPDEs has the same structure as the system of equations for the coefficients of formal power series solutions. Moreover, the structure of this propagator is seemingly universal, i.e., independent of the type of noise. We also introduced new high-order stochastic approximations via Wick-Malliavin series expansions for Gaussian and uniformly distributed noises, and demonstrate convergence as the number of expansion terms increases. Our results are for Burgers and Navier-Stokes (NS) equations but the same approach can be adopted for other nonlinear SPDEs with polynomial nonlinearities and more general noises.

**Multiscale Data-Driven Methods (Caltech leads)** We made significant progress in developing effective numerical methods for solving stochastic partial differential equations and multiscale problems. In particular, we have developed (i) a dynamically bi-orthogonal method for time-dependent SPDEs; (ii) a data-driven stochastic method for multi-query stochastic problems; (iii) a multiscale model reduction method for PDEs with nonseparable multiscale solutions. We have also made progress in deriving a multiscale closure for the 3D incompressible Navier-Stokes equations and in developing data-driven time-frequency analysis by exploiting the intrinsic sparse structure of multiscale data. Below we will give some details on project (i):

We proposed a dynamically bi-orthogonal method (BO) to study time dependent SPDEs. This was inspired by ongoing work in the MURI on dynamically orthogonal expansions (DO). The objective of both BO and DO methods is to exploit some intrinsic sparse structure in the stochastic solution by constructing the sparsest representation of the stochastic solution via a bi-orthogonal basis. These methods essentially track the Karhunen-Loeve expansion dynamically without the need to form the covariance matrix or to compute its eigen-decomposition. In the first part of the work, we derived the dynamically bi-orthogonal formulation for SPDEs, discussed several theoretical issues, such as the dynamic bi-orthogonality preservation and the consistency between the BO formulation and the original SPDE. We also gave some numerical implementation details of the BO methods, including the representation of stochastic basis and techniques to deal with eigenvalue crossing. In the second part, we presented an adaptive strategy to dynamically remove or add modes, performed a detailed complexity analysis, proposed a parallel implementation of DyBO, and discussed various generalizations of this approach. We have applied the BO method to solve the 1D stochastic Burgers equation, 2D incompressible Navier-Stokes equations and the Boussinesq

approximation with Brownian motion forcings. These numerical examples demonstrate that the BO method solves these nonlinear time-dependent SPDEs accurately and efficiently. In subsequent work, the group at Brown derived an exact equivalence between the BO and DO methods and developed a hybrid approach that combines the best computational features of both methods.

## 1.2 Numerical solution of SPDEs

We developed methods to solve the Navier-Stokes and other nonlinear SPDEs in more than 100 dimensions! Several other developments include advances in the generalized polynomial chaos and its variance as well as in Bayesian type methods. In the following, we provide a partial list and in the references we provide all methods we have developed in this MURI.

The *Cornell* PIs focused on sparse Bayesian kernel techniques (relevance vector machines, RVM) for the solution of SPDEs. Each dimension of the multivariate response was modeled using local kernels centered on top of each data point. The missing scale parameters of the kernels were selected by maximizing the joint marginal likelihood of all dimensions of the response. To address issues with high dimensionality and non-informative variance, these algorithms were later extended to weighted mixture of Gaussian processes.

The *Brown* group focused on high-dimensional problems using adaptive ANOVA with applications to the performance analysis of the horn problem, fluid flows and electromagnetic scattering, and also in developing new polynomial chaos methods for white noise. In the context of the horn benchmark, ANOVA was combined with the reduced basis method to enable a similar parametric reduction of the high dimensional problem to allow the development of a certified reduced basis methods for the critical components of systems with many scattering bodies. This allowed for the development of reduced basis methods for problems with many parameters as has been demonstrated for the acoustic horn problem (see below). Another approach to tackling the curse-of-dimensionality is the formulation of PDF equations for colored noise (joint solution-excitation; fractionals PDEs) and solve them using ANOVA or Proper Generalized Decomposition.

The *Caltech* group worked on two methods: Data-Driven Stochastic Multiscale Method (DSM) and Multiscale Multi-Level Monte Carlo Method (MsMLMC), respectively. The second method can be incorporated into the first to boost its applicability and efficiency, especially for tough problems involving randomness and multiscales simultaneously. An important aspect of DSM is the re-usability of the constructed stochastic basis for different deterministic forcing functions. For computational efficiency, a low-rank approximation method is used (developed in compressed sensing) to exploit the low-rank structure of the covariance matrix. Both DSM and MsMLMC have been applied to the horn benchmark with the latter giving up to 100 times speed-up compared to standard MC.

**Uncertainty Quantification for Multiscale PDEs using a Graph Theoretic Approach (Cornell leads)** We developed a probabilistic graphical model based methodology to efficiently perform uncertainty quantification in the presence of both stochastic input and multiple scales. Both the stochastic input and model responses were treated as random variables in this framework. Their relationships were modeled by graphical models which give explicit factorization of a high-dimensional joint probability distribution. The hyperparameters in the probabilistic model were learned using sequential Monte Carlo (SMC) method locally in the graph. Coarse graining (stochastic homogenization) was addressed in a non parametric way using hidden variables in a way that naturally arises within the Bayesian graph theoretic framework. Finally, we made predictions from the probabilistic graphical model using the belief propagation algorithm rather than Monte Carlo integration. Belief propagation has the potential of almost linear scaling in certain applica-

tions. Numerical examples were investigated to show the accuracy and efficiency of the predictive capability of the developed graphical model. Many interesting extensions of this framework were investigated that potentially could lead to a transformative way for UQ in multiscale/multiphysics PDE systems.

**Information Theoretic Coarse Graining: Relative entropy (Cornell leads)** Relative entropy has been shown to provide a principled framework for the selection of coarse-grained potentials. Despite the intellectual appeal of it, its application has been limited by the fact that it requires the solution of an optimization problem with noisy gradients. When using deterministic optimization schemes, one is forced to either decrease the noise by adequate sampling or to resolve to ad hoc modifications in order to avoid instabilities. The former increases the computational demand of the method while the latter is of questionable validity. In order to address these issues and make relative entropy widely applicable, we proposed alternative schemes for the solution of the optimization problem using stochastic algorithms. Cluster expansions are simplified, Ising-like models for binary alloys in which vibrational and electronic degrees of freedom are coarse grained. The usual practice is to learn the parameters of the cluster expansion by fitting the energy they predict to a finite set of ab initio calculations. In some cases, experiments suggest that such approaches may lead to overestimation of the phase transition temperature. We presented a novel approach to fitting the parameters based on the relative entropy framework which, instead of energies, attempts to fit the Boltzmann distribution of the configurational degrees of freedom. We showed how this leads to T-dependent parameters.

**Numerical Methods for High-Dimensional PDF Equations (Brown leads)** In this task we addressed the problem of computing the numerical solution to kinetic partial differential equations involving many phase variables. These types of equations arise naturally in many different areas of mathematical physics, e.g., in particle systems (Liouville and Boltzmann equations), stochastic dynamical systems (FokkerPlanck and DostupovPugachev equations), random wave theory (MalakhovSaichev equations) and coarse-grained stochastic systems (MoriZwanzig equations). We proposed three different classes of new algorithms addressing high-dimensionality: The first one is based on separated series expansions resulting in a sequence of low-dimensional problems that can be solved recursively and in parallel by using alternating direction methods. The second class of algorithms relies on truncation of interaction in low-orders that resembles the BogoliubovBorn-GreenKirkwoodYvon (BBGKY) framework of kinetic gas theory and it yields a hierarchy of coupled probability density function equations. The third class of algorithms is based on high-dimensional model representations, e.g., the ANOVA method and probabilistic collocation methods. A common feature of all these approaches is that they are reducible to the problem of computing the solution to high-dimensional equations via a sequence of low-dimensional problems. The effectiveness of the new algorithms was demonstrated in numerical examples involving nonlinear stochastic dynamical systems and partial differential equations, with up to 120 variables.

**An adaptive hybrid bi-orthogonal/dynamically-orthogonal method for the stochastic Navier-Stokes equations (Brown leads)** A new hybrid methodology for SPDEs was developed based on the dynamically orthogonal (DO) and bi-orthogonal (BO) methods; both approaches are an extension of the Karhunen-Lo'eve (KL) expansion and hence they capture a low-dimensional structure of the solution by tracking the KL expansion of the solution at any given time on-the-fly. It has been shown that DO and BO are equivalent in the sense that one method is an exact reformulation of the other through a matrix differential equation. However, DO suffers numerically when there is a high condition number of the covariance matrix while BO suffers when there is an eigenvalue crossing. To this end, we proposed a unified hybrid framework of the two methods

by utilizing an invertible and linear transformation between them. We also presented an adaptive algorithm to add or remove modes dynamically to better capture the transient behavior. Several numerical examples including the Navier-Stokes equations were presented to illustrate this new adaptive hybrid BO-DO method.

### 1.3 Reduced-Order modeling

#### Certified Basis (MIT leads)

Uncertainty Quantification (UQ) in almost all forms and approaches is performed in a many-query context. It thus follows that UQ is very well suited to the offline-online strategy afforded by model order reduction (MOR) techniques. In the past, application of MOR to UQ has been inhibited by fundamental restrictions in MOR methodology: MOR could treat only a rather limited class of partial differential equations (PDEs); MOR could treat problems characterized by only relatively few parameters. In our MURI effort we have substantially expanded the capabilities of MOR techniques and furthermore proposed new frameworks in which MOR can serve well the goals of UQ. We have also further improved our *a posteriori* error estimators so as to better assess and control the “self-uncertainty” introduced by model order truncation.

Our accomplishments in the area of expanded classes of problems focus on coupled and nonlinear problems, as well as implementations for supercomputers but also deployed platforms. Our accomplishments in the area of increased parameter dimensionality are twofold: related to parameter-domain decomposition, the development of an “*h-p*” reduced basis approximation; related to spatial-domain decomposition, the development of the port-reduced static-condensation reduced-basis element (PR-SCRBE) method for linear problems and also eigenproblems. Our accomplishments in the area of frameworks for uncertainty quantification focus on data assimilation for state estimation. Finally, our accomplishments in the area of *a posteriori* error estimation focus on exact bounds, and on space-time techniques for long-time evolution problems.

We consider these accomplishments in more detail below and more details can be found in the publications of Patera’s group.

*A1. Expanded problem classes.* We have expanded the reduced basis method to consider long-time evolution of nonlinear problems (such as the incompressible Navier-Stokes equations), and also coupled problems (convection-conduction) related to heat exchanger design. We have also expanded the reach of our approach by considering optimized implementations of the reduced basis method on supercomputers and also deployed platforms (for real-time computation).

*A2. Treatment of many parameters.* Most of this effort falls within two methodological thrusts.

The “*h-p*” reduced basis method. This breaks the parameter domain into optimal parameter subdomains and then applies the reduced-basis method on each parameter subdomain. The method permits higher parameter dimensionality due to the smaller domain (hence less rich solution variation) associated with each approximation.

The Port-Reduced Static Condensation Reduced-Basis Element Method (PR-SCRBE). The PR-SCRBE approach is a component-based system synthesis approach which exploits model order reduction at two levels: at the level of ports (the interfaces at which components connect), informed by evanescence arguments; at the level of the component interiors, informed by low-dimensional parametric manifolds. The method can address many parameters, and indeed also topology variations, thanks to the component decomposition and associated divide-and-conquer strategy: we solve many problems with a few parameters rather than one problem with many parameters. The method is also equipped with error estimators both for the port and interior truncations.

*A3. Data Assimilation.* Most of this effort has been focused on the development of a new data assimilation approach, the Parametrized Background Data Weak method, and an associated



rigorous theory of stability and approximation. The distinguishing feature of the method is the effective incorporation of low-dimensional parameter manifolds identified, and approximated, by methods developed within the context of reduced basis methods. The PBDW method is non-intrusive in the sense that the PDE appears only in the offline stage and furthermore provides real-time response in the online stage. The method has been applied within our group to physical systems, in particular to acoustics experiments, with considerable predictive success.

*A4. Improved a posteriori error estimators.* Most of this effort falls within two methodological thrusts.

The first thrust is the development of a formulation in which the reduced-basis error bound is measured with respect to the exact solution of the PDE. In earlier approaches, the reduced-basis error bound is measured relative to a highly refined “truth” finite element approximation. The new approach, proposed and developed by Dr Masa Yano, is preferred not only for the increased rigor, but also because the formulation naturally suggests a simultaneous finite-element reduced-basis adaptive refinement strategy.

The second thrust is the development of improved error estimates for weakly stable evolution problems. In the past, error bounds for weakly stable evolution problems exhibited exponential growth such that only short-time estimates were meaningful. In the new approach, we consider a space-time formulation informed by an optimal inf-sup parameter which considers worst-case growth not from timestep to timestep but rather over the entire time interval and consistent with the governing equation: long-time evolution may thus be pursued.

**Stochastic/Multiscale UQ for Wave Dynamics (Brown leads)** In this research task we focused on three separate but connected efforts. The continued development of certified reduced basis methods in general and with a particular focus on wave problems. We have demonstrated the effectiveness of such methods for a variety of problem types, including parameterized geometries and the use of such models for uncertainty quantification during scattering. A substantial effort has been in the development and application of certified reduced methods for integral equations, including the development of methods that allows the computation of scattering by a collection of scatterers. The challenges associated with the development of reduced models for parametrized models with a high-dimensional parameter space has also been considered. We have developed methods that dramatically accelerate the greedy approximation in the reduced basis development and demonstrated the ability to handle problems with many parameters. In a related work we have shown how to combine reduced models with ANOVA expansions to allow the effective estimation of parametric sensitivity, leading to parameter compression to allow the development of a reduced model for relevant parameters only. A major part of this effort has been devoted to the development and analysis of high-order accurate multi-scale finite element methods. The work is based on a new and more direct high-order multi-scale expression and the analysis confirms optimal behavior. We have considered both the classic Poisson problem as well as completed the first analysis of multi-scale finite element methods for the wave Helmholtz equation. We have also demonstrated how reduced basis methods can be used to reduce the computational overhead associated with heterogeneous multi-scale behavior.

## 1.4 Estimation/Inverse problems

### Bayesian Techniques (Cornell leads)

**Fully Bayesian Uncertainty Quantification Framework/Gaussian Processes with correlated outputs:** Computer codes simulating physical systems usually have responses that consist of a set of distinct outputs (e.g., velocities and pressures) that evolve also in space and time and

depend on many unknown input parameters (e.g., physical constants, initial/boundary conditions etc.). Furthermore, essential engineering procedures such as UQ, inverse problems or design are notoriously difficult to carry out mostly due to the limited simulations available. The aim of this work was to introduce a fully Bayesian approach for treating these problems which accounts for the uncertainty induced by the finite number of observations. Our model was built on a multi-dimensional Gaussian process that explicitly treats correlations between distinct output variables as well as space and/or time. The proper use of a separable covariance function enabled us to describe the huge covariance matrix as Kronecker product of smaller ones leading to efficient algorithms for carrying out inference and predictions. The novelty of this work is the recognition that the Gaussian process model actually defines a posterior probability measure on the function space of possible surrogates for the computer code and the derivation of an algorithmic procedure that allows us to sample it efficiently. We demonstrated how the scheme can be used in uncertainty quantification tasks in order to obtain error bars for the statistics of interest that account for the finite number of observations.

**Sparse Bayesian Techniques.** Multi-output sparse Bayesian techniques (extension of relevance vector machines) that are able to automatically identify the most relevant of a set of basis functions (using either localized kernel functions or an optimal orthogonal polynomial basis). When using an optimal orthogonal polynomials basis, our techniques may be thought as a Bayesian, tree-based extension of generalized Polynomial Chaos (gPC). This combines the optimal convergence of the gPC for smooth functions, with locality capturing discontinuities and the Bayesian framework allowing the quantification of epistemic uncertainty. Our numerical experiments demonstrated that this is a powerful combination. The sparsity of the resulting surrogate: 1) may be intuitively interpretable and 2) are super-fast to evaluate.

**Treed Multi-output Gaussian Process:** We developed an efficient, Bayesian Uncertainty Quantification framework using a novel treed Gaussian process model. The tree is adaptively constructed using information conveyed by the observed data about the length scales of the underlying process. On each leaf of the tree, we utilize Bayesian Experimental Design techniques in order to learn a multi-output Gaussian process. The constructed surrogate can provide analytical point estimates, as well as error bars, for the statistics of interest. We numerically demonstrated the effectiveness of the suggested framework in identifying discontinuities, local features and unimportant dimensions in the solution of SPDEs.

**Solution of Inverse Problems with Limited Forward Solver Evaluations: A Bayesian Perspective.** Solving inverse problems based on computationally demanding forward solvers is ubiquitously difficult since one is necessarily limited to just a few observations of the response surface. This limited information induces additional uncertainties on the posterior distributions. The main contribution of this work is the reformulation of the solution of the inverse problem when the expensive forward model is replaced by a set of simulations. The proposed solution is based on the idea of a Bayesian surrogate that replaces the code. We derived three approximations of the reformulated solution with increasing complexity and fidelity. We demonstrated numerically, that the proposed approximations indeed capture the epistemic uncertainty on the solution of inverse problem induced by the fact that the forward model is replaced by a set of simulations and that they converge to the true solution as the number of simulations is increased.

## 1.5 Robust optimization and control

### Design under Uncertainty (MIT leads)

Under this thrust we have pursued two main research topics: multifidelity methods to accelerate the cost of solving optimization under uncertainty problems, and a goal-oriented approach to inference of distributed parameters. Our multifidelity approaches build on the methods, tools and applications developed in the Model Reduction thrust of the MURI project.

In optimization under uncertainty problems, computing the mean, variance, or other statistics of the high-fidelity model output for every change in the design variables is computationally expensive due to the large number of model evaluations needed. In many practical situations, a low-fidelity model is available to provide useful information about the output of the high-fidelity model at a lower cost. Multifidelity Monte Carlo simulation is a modification of the control variate method that takes advantage of the correlation between the output of the low-fidelity model and the output of the high-fidelity model to reduce the computational cost of uncertainty propagation.

Engineered systems parametrized by distributed quantities represent a significant challenge for state-of-the-art computational methods and inverse problem formulations. An infinite-dimensional parameter is identified to predict output quantities of interest. Goal-oriented inference is the process by which these final outputs are exploited in the inference process. In the linear case, we have shown that the inference algorithm can be suitably modified to improve online efficiency without sacrificing accuracy in prediction of outputs. Our work focused on extending goal-oriented inference to the setting of nonlinear problems. Our work on the deterministic inverse problem formulation has focused on employing error estimation techniques popular in the mesh adaptation community to obtain a parameter estimate that has the correct prediction, but without converging the parameter. In the statistical setting, we extended recent work in Bayesian inference to identify a map from prior predictive to posterior predictive. One then would obtain samples of the posterior predictive directly from applications of the prediction model to prior samples and propagation through the identified map. The required map will generally have many fewer parameters than the analogous map from prior to posterior since it is applied in the prediction space.

## 1.6 Integrated UQ Methodology

In order to demonstrate advances on all five research areas we focused on the specific horn benchmark problem, hence addressing the design of wave-dominated problems under uncertainty. We tested new developments on nonlinear Malliavin calculus, combining reduced basis methods with ANOVA, model validation, on quantifying model uncertainty in inverse problems, on stochastic quantization for the Navier-Stokes equations, on learning techniques, and on stochastic multiscale modeling of materials.

We considered a frequency-domain acoustic planar horn problem first introduced and analyzed in (Udawalpola & Berggren, I. J. Num. Meth. Eng., 73:1571, 2008) in the deterministic framework. The horn consists of a straight waveguide followed by a flare section. The pressure field satisfies a Helmholtz equation with an incoming-wave condition at the waveguide inlet, zero-flow (Neumann) conditions on the horn walls, and a radiation condition for the farfield. We took for our input (stochastic) parameters the flare geometry, wavenumber  $k$ , and impedances  $Z_t, Z_b$  of the top and bottom flare walls; we also must specify a parameter domain. We took for our output the effective reflection coefficient of the horn; the aim is to minimize reflection and hence maximize power transmission. First, we obtained a “truth” finite element (FE) approximation and subsequently we developed a reduced basis (RB) approximation to the FE approximation with corresponding rigorous *a posteriori* error bounds for the difference between the finite element and reduced basis output predictions. The results were reported in a previous progress report (2011).

## 2 Personnel Supported During Duration of Grant

- Faculty: Karniadakis, Hesthaven, Rozovsky, Zabaras, Hou, Patera, Willcox.
- Postdocs (partial support): (Brown) D. Venturi, L. Grinberg, S. Zhang; B. Stamm, P. Gatto, S. Zhang, F. Zeng; (Caltech) Z. Zhang, F-N. Hwang, C. Liu, Z. Shi, Q. Li, G. Luo; (MIT) J. Eftang, D.B.P. Huynh, J.D. Penn, D.J. Knezevic, K. Smetana, T. Tonn, S. Vallaghe, M. Yano, T. Tom, M. Yano, Q. Liao .
- PhD Students (partial support): 14 at Brown; 6 at Cornell; 6 at Caltech; 3 at MIT.

## 3 Honors & Awards

- Hesthaven – SIAM Fellow (2014).
- Hou – AMS Fellow (2012); SIAM Fellow (2009); AAAS (2011).
- Karniadakis, SIAM Fellow (2010); USACM T.J. Oden Medal (2013); R. Kleinmann SIAM award (2015).
- Patera – USACM T.J.R. Hughes Medal (2013); Hans Kupczyk Guest Professorship Award 2010 from the University of Ulm (Germany); Honorary Member, Société de Mathématiques Appliquées et Industrielles (SMAI, France), 2012; Chaire d’Excellence (Senior Research Chair), Fondation Sciences Mathématiques de Paris, France, 2013–2015.
- Willcox and Leo Ng (PhD student with Willcox) received second place in the AIAA Multidisciplinary Analysis and Optimization conference Student Paper competition, for his paper entitled “Multifidelity Uncertainty Propagation for Optimization Under Uncertainty.” (September 2012).

## 4 AFRL Point of Contact

- Jordan, Jennifer L Dr CIV USAF AFMC AFRL/RWM , Energetic Materials Core Technical Competency Lead, Eglin Munitions Directorate visited Cornell and research collaborative plans are under way.
- Willcox discussed multifidelity methods with Ray Kolonay and Ed Alyanak from AFRL.
- Philip Beran, WPAFB, OH 45433, Phone 937-255-665. Visited and gave a talk at MIT in Fall 2010;
- Horie Yasuyuki, CIV USAF AFMC AFRL/RWMER (Yasuyuki.Horie@eglin.af.mil); Dutton, Rolie E Civ USAF AFMC AFRL/RXLMP (Rollie.Dutton@wpafb.af.mil); Cooper, William L Dr CIV USAF AFMC AFRL/RWMWH (william.cooper@eglin.af.mil). A short course on UQ was given by co-PI Zabaras at Eglin AFB and also at General Electric.

## 5 Transitions

- Karniadakis’s group – <http://sourceforge.net/projects/mepcmpackage/> The Multi-Element Probabilistic Collocation Method Package (MEPCMP) is a C++ package which can generate high dimensional, multi-element collocation points based on arbitrary probability density function for the application of Multi-Element Probabilistic Collocation Method (MEPCM).
- Patera’s group – Several technology and software disclosures were made to the MIT Technology Licensing Office during the life of the grant. The MIT Office of Sponsored Research will file the official report on intellectual property. We note here a transition recipient.

Transition Organization: Akselos, Inc <http://www.akselos.com/company.html>

Point of contact within Akselos:

Thomas Leurent, CEO  
Akselos S.A.  
EPFL Innovation Park, Building D  
1015 Lausanne  
Switzerland

E thomas.leurent@akselos.com

Technology Licensed by Akselos: MIT Case No. 15007 “scRBE” by Harriet Li, Dinh Bao Phuong Huynh, David John Knezevic, and Anthony T Patera.

MIT Case No. 15408 “SCRBE V1.1 Software”, by David John Knezevic.

- Hesthaven’s group – Working closely with HyperComp, Inc to transition the use of reduced basis methods and related UQ technology into TEMPUS software. TEMPUS is used widely by AFRL. PhD student X. Zhu spent the summer at HyperComp as an intern working on transitioning reduced order modeling ideas and software into HyperComp Inc.

## 6 Acknowledgement/Disclaimer

This work was sponsored by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-09-1-0613 . The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

## 7 Publications

1. P. Chen, N. Zabaras and I. Bilonis, Uncertainty Propagation using Innie Mixture of Gaussian Processes and Variational Bayesian Inference”, Journal of Computational Physics, Vol. 284, 291-333, 2015.
2. W. Cao, Z. Zhang, G.E. Karniadakis, Time-splitting schemes for fractional differential equations I: smooth solutions, SIAM J. Sci. Comput., 37(4), A1752-A1776, 2015, DOI:10.1137/140996495
3. H. Cho, X. Yang, D. Venturi and G.E. Karniadakis, Algorithms for propagating uncertainty across heterogeneous domains, Accepted for publication in SIAM Journal on Scientific Computing.
4. H. Cho, D. Venturi, G.E. Karniadakis, Numerical methods for high-dimensional probability density function equations, Accepted for publication in JCP.
5. J. S. Hesthaven, S. Zhang and X. Zhu, Reduced basis multiscale finite element methods for elliptic problems, Multi. Model. Simul. 13(1), 316-337, 2015.
6. J. S. Hesthaven, G. Rozza, and B. Stamm, Certified Reduced Basis Methods for Parametrized Partial Differential Equations, Springer Briefs in Mathematics, Springer Verlag. July 2015
7. T. Y. Hou, F. N. Hwang, C. C. Yao. An Iteratively Adaptive Multiscale Finite Element Method, submitted to JCP., 2015.
8. T. Y. Hou, Q. Li, and P. Zhang. Intrinsic Sparse Mode Decomposition for High Dimensional Random Fields, submitted to SIAM/ASA UQ Journal, 2015.

9. T. Y. Hou and P. Liu. A Heterogeneous Stochastic FEM Framework for Elliptic PDEs, JCP, 281, 942-969, 2015,  
DOI: <http://dx.doi.org/10.1016/j.jcp.2014.10.020>.
10. T. Y. Hou and P. Liu. Optimal Local Multi-scale Basis Functions for Linear Elliptic Equations with Rough Coefficients, DCDS-A, accepted, 2015.
11. T. Y. Hou, Z. Shi and P. Tavallali. Sparse Time Frequency Representation and Dynamical Systems, Commun. Math. Sci., 13 (No. 3) 673-694, 2015.
12. T. Y. Hou and P. Liu. Optimal Local Multi-scale Basis Functions for Linear Elliptic Equations with Rough Coefficients, DCDS-A, accepted, 2015.
13. T. Y. Hou and Z. Shi. Extracting a Shape Function for a Signal with Intra-wave Frequency Modulation, Philosophical Transactions A, accepted, 2015.
14. T. Y. Hou and Z. Shi. Sparse Time-Frequency Decomposition Based on Dictionary Adaptation, Philosophical Transactions A, accepted, 2015.
15. C. Liu, Z. Shi, and T. Y. Hou. On the Uniqueness of Sparse Time-Frequency Representation of Multiscale Data, SIAM MMS , 13 (3) , 790-811, 2015. DOI. 10.1137/141002098.
16. T. Y. Hou and P. Liu. A Model Reduction Method for Elliptic PDEs with Random Input Using the Heterogeneous Stochastic FEM Framework, Bulletin of the Institute of Mathematics, Accepted, 2015.
17. Y. Maday, O. Mula, A.T. Patera, and M. Yano, The Generalized Empirical Interpolation Method: Stability Theory on Hilbert Spaces with an Application to the Stokes Equation. Computer Methods in Applied Mechanics and Engineering 287: 310–334, 2015.  
doi: 10.1016/j.cma.2015.01.018.
18. Y. Maday, A.T. Patera, J.D. Penn, and M. Yano, PBDW State Estimation: Noisy Observations; Configuration-Adaptive Background Spaces; Physical Interpretations. CANUM 2014 - 42e Congrès National d'Analyse Numérique, Carry-le-Rouet, France, in ESAIM: Proceedings and Surveys, 50:144–168, 2015. doi:10.1051/proc/201550008.
19. Y. Maday, A.T. Patera, J.D. Penn, and M. Yano, A Parametrized-Background Data-Weak Approach to Variational Data Assimilation: Formulation, Analysis, and Application to Acoustics. International Journal for Numerical Methods in Engineering 102:933–965, 2015. Article first published online August 2014. doi: 10.1002/nme.4747
20. R. Mikulevicius and B.L. Rozovskii, On distribution free Skorokhod-Malliavin calculus, Accepted for publication in the Journal Stochastic Partial Differential Equations: Analysis and Computations, 2015.
21. K. Smetana, A New Certification Framework for the Port Reduced Static Condensation Reduced Basis Element Method. Computer Methods in Applied Mechanics and Engineering, 283:352–383, 2015. doi: 10.1016/j.cma.2014.09.020
22. K. Smetana and A.T. Patera, Optimal Local Approximation Spaces for Component-Based Static Condensation Procedures. SIAM Journal on Scientific Computing (submitted February 2015).

23. F. Song, C. Xu, G.E. Karniadakis, A fractional phase-field model for two-phase flows: Algorithms and Simulations, To Appear CMAME.
24. T. Taddei, J.D. Penn, and A.T. Patera, Experimental A Posteriori Error Estimation by Monte Carlo Sampling of Observation Functionals. *Mathematical Models and Methods in Applied Sciences* (submitted July 2015).
25. S. Vallaghé, D.B.P. Huynh, D Knezevic, TL Nguyen, and AT Patera, Component-Based Reduced Basis for Parametrized Symmetric Eigenproblems. *Advanced Modeling and Simulation in Engineering Sciences*, 2:7, 2015. doi:10.1186/s40323-015-0021-0 (open access paper).
26. M. Yano, A Minimum-Residual Mixed Reduced Basis Method: Exact Residual Certification and Simultaneous Finite-Element Reduced-Basis Refinement. *Mathematical Modelling and Numerical Analysis* (accepted May 2015). doi: dx.doi.org/10.1051/m2an/2015039
27. M. Yano, A Reduced Basis Method with Exact-Solution Certificates for Symmetric Coercive Equations. *Computer Methods in Applied Mechanics and Engineering* 287:290–309, 2015. doi: 10.1016/j.cma.2015.01.003
28. M. Zayernouri, M. Ainsworth, G.E. Karniadakis, Tempered fractional Sturm-Liouville eigenproblems, *SIAM J. Sci. Comput.*, 37(4), A17770A1800, 2015, DOI:10.1137/140985536
29. M. Zayernouri, G.E. Karniadakis, Fractional spectral collocation methods for linear and non-linear variable order FPDEs, *JCP special issue on Fractional PDEs*, 293, 312-338, 2015, doi:10.1016/j.jcp.2014.12.001
30. Z. Zhang, M. Ci and T. Y. Hou. A Multiscale Data-Driven Stochastic Method for Elliptic PDEs with Random Coefficients, *SIAM MMS*, 13 (1), 173-204, 2015. DOI: 10.1137/130948136.
31. M. Zheng and G.E. Karniadakis, Numerical methods for SPDEs with tempered stable processes, *SIAM J. Sci. Comput.* 37(3) A1197-A1217, 2015, DOI:10.1137/140966083
32. M. Zheng, B. Rozovsky, G.E. Karniadakis, Adaptive Wick-Malliavin approximation to Non-linear SPDEs with Discrete Random Variables, *SIAM J. Sci. Comput.* 37(4), A1982-A1990, 2015, DOI:10.1137/140975930
33. P. Chen and N. Zabaras, Uncertainty quantification for multiscale disk forging of polycrystal materials using probabilistic graphical model techniques”, *Computational Materials Science*, Volume 84, 278-292, 2014.
34. H. Cho, D. Venturi, G.E. Karniadakis, Statistical analysis and simulation of random shocks in Burgers turbulence, *Royal Soc. A*, 470, 20140080, 2014, DOI: 10.1098/rspa.2014.0080
35. M. Choi, T. Sapsis and G.E. Karniadakis, On the equivalence of dynamically orthogonal and dynamically bi-orthogonal methods: Theory and numerical simulations. *J. Comput. Phys.*, 270, 1-20, 2014.
36. M. Ci, M. Giles, T. Y. Hou, and Z. Zhang. A multiscale multilevel Monte Carlo method for elliptic PDEs with random coefficients, submitted to *SIAM/ASA J. Uncertainty Quantification*, 2014.
37. M. Ci, T.Y. Hou and Z. Shi, A multiscale model reduction method for partial differential equations, *M2AN*, 48, 449-474, 2014, DOI: 10.1051/m2an/2013115.

38. W. Deng and J. S. Hesthaven, Local discontinuous Galerkin methods for fractional ordinary differential equations, *BIT*, 1-19, 2014.
39. J. S. Hesthaven, B. Stamm and S. Zhang, Efficient greedy algorithms for high-dimensional parameter spaces with applications to empirical interpolation and reduced basis methods, *Math. Model. Numer. Anal.* 48(1), 259-283, 2014.
40. J.S. Hesthaven, S. Zhang and X. Zhu, High-order multiscale finite element methods for elliptic problems, *SIAM Multiscale Model Simul* 12(2), 650-666, 2014.
41. T.Y. Hou, Z. Shi and P. Tavalali, Convergence of a data-driven time-frequency analysis method, *Applied and Comput. Harmonic Analysis*, 37(2), 235-270, 2014, Doi: 10.1016/j.acha.2013.12.004.
42. D.B.P. Huynh, A Static Condensation Reduced Basis Element Approximation: Application to three-dimensional acoustic muffler analysis. *International Journal of Computational Methods*, 11(3):1343010 (16 pages), 2014. doi: 10.1142/S021987621343010X.
43. J. Kristensen and N. Zabaras, Bayesian uncertainty quantification in the evaluation of alloy properties with the cluster expansion method”, *Computer Physics Communications*, Vol. 185, 2885-2892, 2014.
44. Y. Maday, A.T. Patera, J.D. Penn, and M. Yano, A Parametrized-Background Data-Weak Approach to Variational Data Assimilation: Formulation, Analysis, and Application to Acoustics. *International Journal for Numerical Methods in Engineering* 102:933–965, 2015. Article first published online August 2014. doi: 10.1002/nme.4747
45. L. Ng and K. Willcox, Aircraft conceptual design under uncertainty, In *Proceedings of the 10th AIAA Multidisciplinary Design Optimization Conference*, Baltimore, MD, January 2014. Submitted to *Journal of Aircraft*.
46. B. Peherstorfer, D. Butnaru, K. Willcox and H. Bungartz, Localized discrete empirical interpolation methods, Accepted for publication, *SIAM Journal on Scientific Computing*.
47. P. Tavallali, T. Y. Hou, and Z. Shi. Extraction of Intrawave Signals Using the Sparse Time-Frequency Representation Method, *SIAM MMS*, 12 (No. 4), 1458-1493, 2014. DOI: 10.1137/140957767.
48. K. Urban and A.T. Patera, An Improved Error Bound for Reduced Basis Approximation of Linear Parabolic Problems, Submitted to *Mathematics of Computation*, 83(288), 1599-1615, 2014 doi: 10.1090/S0025-5718-2013-02782-2.
49. S. Vallaghé and A.T. Patera, The Static Condensation Reduced Basis Element Method for a Mixed-Mean Conjugate Heat Exchanger Model. *SIAM Journal on Scientific Computing* 36(3), B294–B320, 2014. doi: 10.1137/120887709 (Due to a clerical error, this paper does not include proper acknowledgment to sponsors. On our website [augustine.mit.edu](http://augustine.mit.edu) we include this errata and cite acknowledgments to sponsors.)
50. D. Venturi and G.E. Karniadakis, Convolutionless Nakajima-Zwanzig equations for stochastic analysis in large scale simulations, *Royal Society A*, 470, 20130754, 2014.
51. J. Wan and N. Zabaras, Stochastic Input Model generation using Bayesian Network Learning”, *Journal of Computational Physics*, Vol. 272, 664-685, 2014.



52. Q. Xu and J. S. Hesthaven, Stable multi-domain spectral penalty methods for fractional partial differential equations, in *J. Comput. Phys.* 257, 241-258, 2014.
53. Q. Xu and J. S. Hesthaven, Discontinuous Galerkin method for fractional convection-diffusion equations, *SIAM J. Numer. Anal.* 52(1), 405-423, 2014.
54. M Yano, A Space-Time Petrov-Galerkin Certified Reduced Basis Method: Application to the Boussinesq Equations. *SIAM Journal on Scientific Computing* 36(1):A232–A266, 2014. doi: 10.1137/120903300
55. M. Yano, A.T. Patera and K. Urban, A space-time *hp*-interpolation-based certified reduced basis method for Burgers' equation, *Mathematical Models and Methods in Applied Sciences*, 24(9), 2014, doi: 10.1016/j.crma.2013.10.034.
56. M. Zayernouri and G.E. Karniadakis, Exponentially accurate spectral and spectral element methods for fractional ODEs, *J. Comput. Phys.*, 257, Part A, 460-480, 2014.
57. M. Zayernouri and G.E. Karniadakis, Fractional spectral collocation methods, *SIAM J. Sci. Comput.*, 36(1 A40-A62, 2014.
58. M. Zayernouri and G.E. Karniadakis, Fractional spectral collocation methods for linear and nonlinear variable order FPDEs, Submitted to *J. Comput. Physics*, Special Issue on Fractional PDEs, 2014.
59. M. Zheng and G.E. Karniadakis, Numerical Methods for SPDEs with tempered stable processes, Submitted to *SIAM J. Sci. Comput.* 2014.
60. I. Bilonis and N. Zabaras, Solution of inverse problems with limited forward solver evaluations: a fully Bayesian framework, *Inverse problems* (under review) 2013.
61. I. Bilonis and N. Zabaras, A stochastic optimization approach to coarse-graining using a relative framework, *The Journal of Chemical Physics*, 138(4), 044313, 2013.
62. I. Bilonis, N. Zabaras, B.A. Konomi, and G. Lin, Multi-output separable Gaussian process: Towards an efficient, fully Bayesian paradigm for uncertainty quantification, *J. of Comput. Physics*. 241, 212-239, 2013.
63. W. Cao, Z. Zhang and G.E. Karniadakis, Numerical methods for stochastic delay differential equations via the Wong-Zakai approximation, Submitted to *SIAM J. Sci. Comput.* 2013.
64. P. Chen and N. Zabaras, Adaptive locally weighted projection regression method for uncertainty quantification", *Communications in Computational Physics (CiCP)*, 14(4), 851-878, 2013, doi: 10.4208/cicp.060712.281212a.
65. P. Chen and N. Zabaras, A nonparametric belief propagation method for uncertainty quantification with applications to flow in random porous media", *Journal of Computational Physics*, Vol. 250, 616-643, 2013.
66. M. Cheng, T. Y. Hou, and Z. Zhang, A Dynamically Bi-Orthogonal Method for Time-Dependent Stochastic Partial Differential Equations I: Derivation and Algorithms', *J. Comput. Phys.*, 242, 843-868, 2013, DOI: 10.1016/j.jcp.2013.02.033.

67. M. Cheng, T. Y. Hou, and Z. Zhang, A Dynamically Bi-Orthogonal Method for Time-Dependent Stochastic Partial Differential Equations II: Adaptivity and Generalizations, *J. Comput. Phys.* 242, 753-776, 2013, DOI: 10.1016/j.jcp.2013.02.020.
68. M. Cheng, T.Y. Hou, M. Yan and Z. Zhang, A data-driven stochastic method for elliptic PDEs with Random coefficients, *SIAM/ASA J. for Uncertainty Quantification* 1, 452-493, 2013, DOI: 10.1137/130913249.
69. H. Cho, D. Venturi and G. Karniadakis, Adaptive discontinuous Galerkin method for response-excitation PDF equations, *SIAM J. Sci. Comput.* 35(4), B90-B911, 2013.
70. H. Cho, D. Venturi, and G.E. Karniadakis, Karhunen-Loeve expansion for multi-correlated stochastic processes, *Probabilistic Engineering Mechanics*, 34, 147-167, 2013.
71. M. Choi, T. Sapsis, and G.E. Karniadakis, A convergence study for the SPDEs using combined polynomial chaos and dynamically orthogonal schemes, *J. Comp. Phys.* 245, 281-301, 2013.
72. W. Deng and J. S. Hesthaven, Local discontinuous Galerkin methods for fractional diffusion equations, *Math. Model. Numer. Anal.* 47(6), 1845-1864, 2013.
73. Y. Efendiev, J. Galvis and T.Y. Hou, Generalized multiscale finite element methods (GMs-FEM), *J. Comput. Phys.*, 251, 116-135, 2013.
74. J.L. Eftang and A.T. Patera, A port-reduced static condensation reduced basis element method for large component-synthesized structures: Approximation and a posteriori error estimation. *Advanced Modeling and Simulation in Engineering Sciences (SpringerOpen)* 1:3, 2013.
75. J.L. Eftang, D.B.P. Huynh, D.J. Knezevic, E.M. Rnquist and A.T. Patera, Adaptive Port Reduction in Static Condensation: Proceedings of the 7th Vienna Conference on Mathematical Modelling - MATHMOD2012 ) DOI: 10.3182/20120215-3-AT-3016.00123, vol. 7, part 1, pp. 695-699, 2013.
76. J.S. Hesthaven, B. Stamm, and S. Zhang, Efficient greedy algorithms for high-dimensional parameter spaces with application so empirical interpolation and reduced basis methods, *Mathematical Modeling and Numerical Analysis*, Published online by Cambridge University Press, July 16, 2013, DOI: <http://dx.doi.org/10.105/m2an/2013100>
77. T. Y. Hou and Z. Shi, Data-driven Time-Frequency Analysis, *Applied and Comput. Harmonic Analysis*, 35(2), 284, 308, 2013.  
<http://dx.doi.org/10.1016/j.acha.2012.10.001>.
78. T.Y. Hou, X. Hu and F. Hussain, Multiscale modeling of incompressible turbulent flows, *J. Comput. Phys.*, 232, 383, 396, 2013.  
<http://dx.doi.org/10.1016/j.jcp.2012.08.029>.
79. T. Y. Hou and Z. Shi. Sparse Time Frequency Representation of Nonlinear and Nonstationary Data, *Science China, Mathematics*, 56 (No. 12), 2489-2506, 2013. DOI: 10.1007/s11425-013-4733-7.
80. D.B.P. Huynh, D.J. Knezevic and A.T. Patera, A Static Condensation Reduced Basis Element Method: Complex Problems: *Computer Methods In Applied Mechanics and Engineering*, 259(1), 197-216, 2013, DOI: 10.1016/u.cma.2013.02.013.

81. D.B.P. Huynh, D.J. Knezevic and A.T. Patera, A Static Condensation Reduced Basis Element Method: Approximation and A Posteriori Error Estimation, *Mathematical Modeling and Numerical Analysis*, 47(1), 213-251, 2013, DOI: 10.1051/m2an/2012022.
82. J. Kristensen, I. Bilonis and N. Zabaras, Relative entropy as model selection tool in cluster expansion, *Phys. Rev. B*, 87, 174112, 2013.
83. C. Lieberman and K. Willcox, Nonlinear goal-oriented Bayesian inference: Application to carbon capture and storage, Submitted to *SIAM J. Sci. Comput.*, 2013.
84. C. Lieberman and K. Willcox, Goal-oriented inference: Approach, linear theory, and application to advection-diffusion, *SIAM Review*, 55(3), 493-519, 2013.
85. C. T. Miller, C. N. Dawsonb, M. W. Farthing, T. Y. Hou, J.F. Huange, C. E. Kees, C.T. Kelley, H.P. Langtangen Numerical Simulation of Water Resources Problems: Models, Methods, and Trends', *Advances in Water Resources*, 51, 405-437, 2013. DOI: 10.1016/j.advwatres.2012.05.008.
86. L. Ng, and K. Willcox, Aircraft Conceptual Design Under Uncertainty, 10th AIAA Multidisciplinary Design Optimization Conference, DOI:10.2514/6.2014-0802.
87. L. Ng and K. Willcox, Multifidelity approaches for optimization under uncertainty, Submitted to *International Journal for Numerical Methods in Engineering*, 2013.
88. B. Peherstofer, D. Butnaru, K. Willcox and H. Bungartz, Localized Discrete Empirical Interpolation Method, *Aerospace Computational Design Lab Technical Report ACDL-TR-2013*, Submitted to *SIAM Journal on Scientific Computing*, 2013.
89. L. Ng and K. Willcox, Aircraft Conceptual Design under Uncertainty, Submitted to *AIAA 2014 SciTech Conference*.
90. B. Rozovsky and X. Wan, Wick-Malliavin approximation of elliptic problems with long-normal random coefficients, Submitted 2013.
91. K. Urban and A.T. Patera, An Improved Error Bound for Reduced Basis Approximation of Linear Parabolic Problems. *Mathematics of Computation*, 83(288):1599–1615, 2014 (published online October 2013). doi: 10.1090/S0025-5718-2013-02782-2.
92. S. Vallaghe, D.B.P. Huynh, D. Knezevic and A.T. Patera, Component-based reduced basis for eigenproblems Submitted July 2013 to *Computers & Structures*.
93. D. Venturi, D.M. Tartakovsky, A.M. Tartakovsky and G.E. Karniadakis, Exact PDF equations and closure approximations for advective-reactive transport, *J Comput. Phys.*, 243, 323-343, 2013.
94. D. Venturi and G.E. Karniadakis, Goal-oriented probability density function methods for stochastic dynamic al systems, Submitted to *PNAS*, 2013.
95. D. Venturi, R. Mikulevicius, B. Rozovskii, and G.E. Karniadakis, Wick-Malliavin approximation to nonlinear stochastic PDEs: Analysis and simulations, Accepted for publication in *Proceedings of the Royal Society A*, 2013.
96. J. Wan and N. Zabaras, A probabilistic graphical model approach to stochastic multiscale partial differential equations, *J. Comput. Physics*, 250, 477-510, 2013.

97. X. Yang, and G.E. Karniadakis Reweighted minimization method for stochastic elliptic differential equations, *J. Comput. Phys.*, 248, 87-108, 2013.
98. M. Yano and A.T Patera, A space-time variational approach to hydrodynamic stability theory. *Proceedings of the Royal Society A*, 469(2155), article number 20130036, July 2013. doi: 10.1098/rspa.2013.0036
99. M. Yano, A reduced basis method with exact-solution certificates for symmetric coercive equations, Submitted in November 2013 to *Computer Methods in Applied Mechanics and Engineering*.
100. M. Yano, A space-time Petrov-Galerkin certified reduced basis method: Application to the Boussinesq equations, *SIAM Journal on Scientific Computing*, Accepted November 2013.
101. M Yano, JD Penn, and AT Patera, A Model-Data Weak Formulation for Simultaneous Estimation of State and Model Bias. *Comptes Rendus Mathematique*, 351(23-24):937–941, 2013. doi: 10.1016/j.crma.2013.10.034.
102. N. Zabaras, Adaptive local weighted projection regression method for uncertainty quantification, *Communications in Computational Physics* (in press) 2013.
103. M. Zayernouri, S-W. Park, D. Tartakovsky and G.E. Karniadakis, Stochastic smoothed profile method for modeling random roughness in flow problems, *CMAME*, 263, 99-112, 2013.
104. M. Zayernouri, and G.E. Karniadakis, Fractional Sturm-Liouville eigen-problems: Theory and numerical approximation, *J. Comput. Phys.*, 252, 495-517, 2013.
105. M. Zayernouri, W. Cao and G.E. Karniadakis, Spectral and discontinuous spectral element methods for fractional delay equations, Submitted to *SIAM J. Sci. Comput.*, 2013.
106. item M. Zayernouri, I. Alvey and G.E. Karniadakis, Discontinuous Petrov-Galerkin methods for time and space-fractional advection equations, Submitted to *SIAM J. Sci. Comput.* 2013.
107. M. Zayernouri, M. Ainsworth and G.E. Karniadakis, A unified Petrov-Galerkin spectral method for fractional PDEs, Submitted to *CMAME*, 2013.
108. Z. Zhang, X. Hu, T.Y. Hou, G. Lin and M. Yan, An adaptive ANOVA-based data-driven stochastic method for elliptic PDEs with random coefficients, Submitted to *Communications in Computational Physics*, 2013.
109. Z. Zhang, M. Ci and T.Y. Hou, A multiscale-data-driven stochastic method for elliptic PDEs with random coefficients, Submitted to *SIAM Multiscale Modeling and Simulation*, 2013.
110. Z. Zhang, M. Ci and T.Y. Hou, A multiscale multilevel Monte Carlo method for elliptic PDEs with random coefficients, In preparation, 2013.
111. Z. Zhang, X. Yang, G. Lin and G.E. Karniadakis, Numerical solution for the Stratnovich-and Ito-Euler equations: Application to the stochastic piston problem, *J. Comput. Phys.*, 236, 15-27, 2013.
112. Z. Zhang, B. Rozovskii and G.E. Karniadakis, The method of lines for stochastic differential equations driven by white noise: A spectral approach, In preparation 2013.

113. Z. Zhang, M.V. Tretyakov, B. Rozovskii and G.E. Karniadakis, Wiener chaos vs stochastic collocation methods for linear advection-diffusion-reaction equations with multiplicative white noise, Submitted to SIAM J. on Numerical Analysis, 2013.
114. Z. Zhang, M.V. Tretyakov, B. Rozovskii and G.E. Karniadakis, A recursive sparse grid collocation method for differential equations with white noise, Submitted to SIAM J. Sci. Comput. 2013.
115. M. Zheng, X. Wan, and G.E. Karniadakis, Adaptive-multi-element polynomial chaos with discrete measure: Algorithms and application to SPDEs, Submitted to Applied Numerical Mathematics, 2013.
116. I. Bilonis and N. Zabaras, Multi-output local Gaussian process regression: Application to uncertainty quantification, J. Comp. Phys., 231, 5718-5746, 2012.
117. I. Bilonis and P. S. Koutsourelakis. Free energy computations by minimization of Kullback-Leibler divergence: An efficient adaptive biasing potential method for sparse representation, Journal of Computational Physics, 231, 3849-3870, 2012.
118. I. Bilonis and N. Zabaras, Multidimensional adaptive relevance vector machines for Uncertainty Quantification”, SIAM Journal for Scientific Computing, 34(6), B881-B908, 2012.
119. Y. Chen, J.S. Hesthaven, Y. Maday, J. Rodriguez, and X. Zhu. Certified reduced methods for electromagnetic scattering and radar cross section estimation, Comput. Methods Appl. Mech. Engin., 233, 92-108, 2012.
120. J.L. Eftang, D.B.P. Huynh, D.J. Knezevic, E.M. Rønquist, and A.T. Patera, Adaptive Port Reduction in Static Condensation. Proceedings of 7th Vienna Conference on Mathematical Modelling (MATHMOD 2012), eds. I Troch and F Breitenecker, Mathematical Modelling, 7(1):695–699, 2012. doi: 10.3182/20120215-3-AT-3016.00123.
121. J.L. Eftang, D.B.P. Huynh, D.J. Knezevic, and A.T. Patera. A two-step certified reduced basis method, Journal of Scientific Computing, 51(1), 28-58, (doi: 10.1007/s10915-011-9494-2) 2012.
122. J.L. Eftang and B. Stamm, Parameter multi-domain ”hp” empirical interpolation, International Journal for Numerical Methods in Engineering, 90(4), 412-428, doi: 10.1002/nme.3327, 2012.
123. M. Ganesh, J.S. Hesthaven, and B. Stamm. A reduced basis method for multiple electromagnetic scattering in three dimensions, J. Comput. Phys. 231(23), 7756-7779, 2012.
124. J.S. Hesthaven, B. Stamm, and S. Zhang. Certified reduced basis methods for the electric field integral equation SIAM J. Sci. Comput., 34(3), A1777-A1799, 2012.
125. D.B.P. Huynh, D.J Knezevic, and A.T Patera. Certified reduced basis model characterization: a frequentistic uncertainty framework, Computer Methods in Applied Mechanics and Engineering, 201-204, 13-24 (doi: 10.1016/j.cma.2011.09.011) 2012.
126. C.Y. Lee and Boris Rozovskii. On stochastic Navier-Stokes equations driven by stationary white noise, In: Malliavin Calculus and Stochastic Analysis, F. Viends et al, Editors, pp. 219-250.

127. C. Lieberman and K. Willcox, Goal-oriented inference: Approach, linear theory, and action to advection-diffusion, *SIAM J. Sci. Comput.* 34(4), 1880-1904, 2012.
128. S. Lototsky and B.L. Rozovskii, D. Selesi. On generalized Malliavin calculus, *J. Stochastic Processes and Applications*, 122, 808-843, 2012.
129. R. Mikulevicius, B. L. Rozovsky, On unbiased stochastic Navier-Stokes equation, On unbiased stochastic Navier-Stokes equations, *Probab. Theory Related Fields*, 154, 787-834, 2012.
130. L. Ng, D.B.P. Huynh and K. Willcox, Multifidelity uncertainty propagation for optimization under uncertainty, 12th AIAA Aviation Technology, Integration and Operations (ATIO) Conference and 14th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, Chapter DOI: 10.2514/6.2012-5602, September 17-19 2012.
131. A.T. Patera and E.M. Ronquist, Regression on parametric manifolds: Estimation of spatial fields, functional outputs, and parameters from noisy data, *CR Acad. Sci. Paris, Series 1*, 350(9-10), 543-547, 2012, doi:10.1016/j.crma.2012.05.002
132. K. Urban and A.T. Patera, A new error bound for reduced basis approximation of parabolic partial differential equations, *CR Acad. Sci. Paris Series 1*, 350(3-4), 203-207, 2012, doi:10.1016/j.crma.2012.01.026.
133. D. Venturi, T. Sapsis, H. Cho, and G.E. Karniadakis. A computable evolution equation for the joint response-excitation probability density function of stochastic dynamical systems, *Proc. Roy. Soc. A.*, 468, 759-793 2012 (doi: 10.1098/rspa.2011.0186).
134. D. Venturi and G.E. Karniadakis. New evolution equations for the joint response-excitation probability density function of stochastic solutions to first-order nonlinear PDEs, *Comp. Physics*, 231, 7450-7474, 2012.
135. D. Venturi and G.E. Karniadakis. Differential constraints for the probability density function of stochastic solutions to the wave equation, *Int. J. Uncertainty Quantification*, 2(3), 195-213, 2012.
136. D. Venturi, M. Choi, and G.E. Karniadakis. Supercritical quasi-conduction states in stochastic Rayleigh Benard convection, *Int. J. of Heat & Mass Transfer*, 55(13-14), 3732-3743, 2012.
137. B. Wen and N. Zabaras. Investigating variability of fatigue indicator parameters of two-phase nickel-based superalloy microstructures, *Computational Materials Science*, Vol. 51(1), pp. 455-481, 2012.
138. B. Wen and N. Zabaras, A multiscale approach for model reduction of random microstructures, *Computational Materials Science*, Vol. 63, pp. 269-285, 2012.
139. X. Yang, M. Choi, G. Lin, and G.E. Karniadakis. Adaptive ANOVA decomposition of stochastic incompressible and compressible flows, *J. Comp. Phys.*, 231, 1587-1614, (doi: 10.1016/j.jcp.2011.10.028) 2012.
140. Z. Zhang, M. Choi, and G.E. Karniadakis. Error estimates for the ANOVA method with polynomial chaos interpolation: Tensor product functions, *SIAM J. Sci. Comp.*, 34(2) A1165-A1186, 2012.

141. Z. Zhang, B. Rozovskii, M.V. Tretyakov, G.E. Karniadakis, A multi-stage Wiener chaos expansion method for stochastic advection-diffusion-reaction equations, *SIAM J. Sci. Comput.* 34(2), A914-A936, 2012.
142. J.L. Eftang, D.J. Knezevic, and A.T. Patera. An hp certified reduced basis method for parametrized parabolic partial differential equations, *Mathematical and Computer Modelling of Dynamical Systems*, 17(4), 395-422, (doi:10.1080/13873954.2011.547670) 2011.
143. J.L. Eftang, AT Patera, and EM Rnquist, An “hp” Certified Reduced Basis Method for Parametrized Parabolic Partial Differential Equations. JS Hesthaven and EM Rnquist (eds.), *Spectral and High Order Methods for Partial Differential Equations, Lecture Notes in Computational Science and Engineering*, 76:179–187, 2011. doi: 0.1007/978-3-642-15337-2-15
144. B. Fares, J.S. Hesthaven, Y. Maday, and B. Stamm. The reduced basis method for the electric field integral equation, *J. Comput. Phys.* 230(14), 5532-5555, (doi: 10.1016/j.jcp.2011.03.023) 2011.
145. Z.Gao and J.S. Hesthaven. Efficient solution of ordinary differential equations with high-dimensional parametrized uncertainty, *Comm. Comput. Phys.* 10(2), 253-278, 2011.
146. T. Y. Hou, Z. Shi, and S. Wang. The dual role of convection in 3D Navier-Stokes equations, to appear in *Proceedings of 2011 Congress of Foundation of Computational Mathematics*, Budapest, *Lecture Notes in Applied Mathematics*, 2011.
147. X. Hu, G. Lin, T.Y. Hou and P. Yan. An adaptive ANOVA-based data-driven stochastic method for elliptic PDE with random coefficient, submitted 2011.
148. D.B.P. Huynh, D.J. Knezevic, and A.T Patera. A Laplace transform certified reduced basis method; application to the heat equation and wave equation, *CR Acad Sci Paris Series I*, 349(7-8):401-405, 2011, doi:10.1016/j.crma.2011.02.003
149. D.B.P. Huynh, D.J Knezevic, J.W Peterson, and A.T Patera. High-fidelity real-time simulation on deployed platforms, *Computers and Fluids*, 43(1):74-81 (doi:10.1016/j.compfluid.2010.07.007) 2011.
150. D.J Knezevic and J.W. Peterson. A High-performance parallel implementation of the certified reduced basis method, *Comput. Methods Appl. Mech. Engrg.* 200(13-16), 1455-1566, (doi: 10.1016/j.cma.2010.12.026) 2011.
151. D.J. Knezevic, N.C Nguyen, and A.T Patera. Reduced basis approximation and a posteriori error estimation for the parametrized unsteady Boussinesq equations, *Mathematical Models and Methods in Applied Sciences*, 21(7), 1415-1442, 2011, (doi: 10.1142/S0218202511005441)
152. P.S. Koutsourelakis and E. Bilonis. Scalable Bayesian reduced-order models for simulating high-dimensional multiscale dynamical systems, *SIAM Multiscale Modeling & Simulation*, 9(1), 449-485, 2011.
153. M. Liu, Z. Gao and J.S. Hesthaven. Adaptive sparse grid algorithms with applications to electromagnetics scattering under uncertainty, *Appl. Numer. Math.* 61(1), 24-37, (doi:10.1016/j.apnum.2010.08.2011).

154. X. Ma and N. Zabaras. Kernel principal component analysis for stochastic input model reduction, *Journal of Computational Physics*, Vol. 230(19), 7311-7331, 2011.
155. X. Ma and N. Zabaras. A stochastic mixed finite element heterogeneous multiscale method for flow in porous media, *Journal of Computational Physics*, Vol. 230(12), 4696-4722, 2011.
156. J. Park, B. Rozovsky, and R. Sowers. Efficient nonlinear filtering of a singularly perturbed stochastic hybrid system, *London Math. Society J. of Computation and Mathematics*, 2011, (doi: 10.1112/S146115701000029X).
157. D. Venturi. A fully symmetric nonlinear biorthogonal decomposition theory for random fields, *Physica D*, 240(4-5), 415-25, 2011, (doi: 10.1016/j.physd.2010.10.005) 2011.
158. J. Wan and N. Zabaras. A Bayesian approach to multiscale inverse problems using a sequential monte carlo method, *Inverse Problems*, Vol. 27(10), 2011.
159. B. Wen, Z. Li and N. Zabaras. Thermal response variability of random polycrystalline microstructures, *Communications in Computational Physics*, Vol. 10(3), 607-634, 2011.
160. J.L. Eftang, M.A. Grepl, and A.T. Patera. A posteriori error bounds for the empirical interpolation method, *CR Acad Sci Paris Series I*, 348(9-10): 575-579, 2010, doi:10.1016/j.crma.2010.03.004,
161. J.L. Eftang, A.T. Patera, and E.M. Rnquist. An “hp” certified reduced basis method for parametrized elliptic partial differential equations, *SIAM Journal on Scientific Computing*, 32(6), 3170-3200, 2010, (doi: 10.1137/090780122).
162. Z. Gao and J.S. Hesthaven. On ANOVA expansions and strategies for choosing the anchor point, *Appl. Math. and Comp.* 217, 3274-3285, 2010, (doi:10.1016/j.amc.2010.08.061).
163. B. Kouchmeshky and N. Zabaras. Microstructure model reduction and uncertainty quantification in multiscale deformation processes, *Computational Materials Science*, 48:213–227, 2010.
164. C.-Y. Lee and B. Rozovskii. A stochastic finite element method for stochastic parabolic equations driven by purely spatial noise, *Communications on Stochastic Analysis*, 4(2), 271-297, 2010.
165. C.-Y. Lee, B. Rozovskii, and H. M. Zhou. Randomization of forcing in large systems of PDE for improvement of energy estimates, *SIAM J. Multiscale Modeling and Simulation*, 8(4), 1419-1438, 2010.
166. Z. Li, B. Wen, and N. Zabaras. Computing mechanical response variability of polycrystalline microstructures through dimensionality reduction techniques, *Computational Materials Science*, Vol. 49(3), pp. 568-581, 2010.
167. S. Lototsky, B. Rozovskii, and X. Wan. Elliptic equations of higher stochastic order, *J. Math. Modeling and Numerical Anal*, 44(5), 1135-1153, 2010.
168. X. Ma and N. Zabaras. An adaptive high-dimensional stochastic model representation technique for the solution of stochastic PDEs, *J. Comp. Phys.*, 229:3884–3915, 2010.



- 169. X. Ma and N. Zabararas. An adaptive high-dimensional stochastic model representation technique for the solution of stochastic PDEs, *Journal of Computational Physics*, Vol. 229(10), 3884-3915, 2010.
- 170. D. Venturi, X. Wan, and G.E. Karniadakis. Stochastic bifurcation analysis of Rayleigh-Benard Convection, *Journal of Fluid Mechanics* 650, 391-413, 2010.
- 171. Z. Zhang, M. Choi and G.E. Karniadakis. Anchor points matter in ANOVA decomposition, *Proceedings of ICOSAHOM'09*, Springer, eds. E. Ronquist & J. Hesthaven, 2010.

1.

**1. Report Type**

Final Report

**Primary Contact E-mail****Contact email if there is a problem with the report.**

George\_Karniadakis@brown.edu

**Primary Contact Phone Number****Contact phone number if there is a problem with the report**

401-863-1414

**Organization / Institution name**

Brown University

**Grant/Contract Title****The full title of the funded effort.**

Multi-scale Fusion of Information for Uncertainty Quantification and Management in Large-scale Simulations

**Grant/Contract Number****AFOSR assigned control number. It must begin with "FA9550" or "F49620" or "FA2386".**

FA9550-09-1-0613

**Principal Investigator Name****The full name of the principal investigator on the grant or contract.**

George Em Karniadakis

**Program Manager****The AFOSR Program Manager currently assigned to the award**

Dr. Jean-Luc Cambier

**Reporting Period Start Date**

09/01/2009

**Reporting Period End Date**

08/31/2015

**Abstract**

We developed an integrated methodology for uncertainty quantification (UQ) that proceeds from initial problem definition to engineering applications. Towards this goal, we worked on five research areas: (1) Mathematical analysis of Stochastic Partial Differential Equations (SPDEs) and multiscale formulation; (2) Numerical solution of SPDEs; (3) Reduced-Order modeling; (4) Estimation/Inverse problems; and (5) Robust optimization and control. This work sets the mathematical foundations of Uncertainty Quantification methods used by many diverse communities in computational mechanics, fluid dynamics, plasma dynamics, and materials science. We have pioneered methods for efficient high-dimensional representations of stochastic processes, established Wick-Malliavin approximation for nonlinear SPDEs, theoretical error estimates for multiscale parametric and stochastic PDEs, a new approach to design of experiment and UQ on parametric manifolds, multi-fidelity optimization-under-uncertainty, a data-driven Bayesian framework and probabilistic graphical models for UQ, and information-based coarse graining methods. We have also demonstrated an integration of our UQ methodology and all five areas for a benchmark problem. We have published more than 150 papers in top mathematical journals, obtained one patent (MIT), and have established one software company (MIT).

DISTRIBUTION A: Distribution approved for public release

Specific research highlights include:

Mathematical Theory: Quantization-renormalization of SPDEs; New evolution equations for joint-pdf of SPDEs; Nonlinear Malliavin calculus.

Reduced Basis Methods (RBM): Integral equations and multi-scattering problems; Robust design, parameter estimation, and model uncertainty.

Adaptive ANOVA: Convergence theory; Parameter compression and RBM; Fluid flows, porous media, multi-scattering.

Bayesian Framework: coarse-graining; Active learning + SPDEs; Adaptive SMC, dependent random variables, Model uncertainty in inverse problems.

Numerical SPDEs: Data-driven stochastic multiscale method, Multiscale multilevel MC, Probabilistic graphical models, Long-time integrators of SPDEs.

Software: MEPCM library (polynomial chaos + ANOVA); RBM libraries - RBOOMIT, RAppMIT; Akselos, Inc; Random poly-crystals – RPCrystal; TEMPUS/Hypercomp.

In addition, we have extended the original proposal to introduce new concepts as follows:

Stochastic Delay ODEs & PDEs

Fractional PDEs – First Symposium on June 3-5, 2013 (Newport, RI)

Reduced Basis for design of experiments

Multiscale high-order FEM and Reduced Basis

A control-theoretic approach to “inference for prediction”

UQ for polycrystals using microstructure images and FFT

Probabilistic graphical models for UQ

Bayesian surrogates to compute epistemic uncertainty

## **Distribution Statement**

This is block 12 on the SF298 form.

Distribution A - Approved for Public Release

## **Explanation for Distribution Statement**

If this is not approved for public release, please provide a short explanation. E.g., contains proprietary information.

## **SF298 Form**

Please attach your SF298 form. A blank SF298 can be found [here](#). Please do not password protect or secure the PDF. The maximum file size for an SF298 is 50MB.

[AFD-070820-035.pdf](#)

**Upload the Report Document. File must be a PDF. Please do not password protect or secure the PDF. The maximum file size for the Report Document is 50MB.**

[FINAL\\_FA9550-09-1-0613.pdf](#)

**Upload a Report Document, if any. The maximum file size for the Report Document is 50MB.**

**Archival Publications (published) during reporting period:**

**Changes in research objectives (if any):**

**Change in AFOSR Program Manager, if any:**

Old Program Manager: Dr. Fariba Fahrool

DISTRIBUTION A: Distribution approved for public release

New Program Manager: Dr. Jean-Luc Cambier

**Extensions granted or milestones slipped, if any:**

**AFOSR LRIR Number**

**LRIR Title**

**Reporting Period**

**Laboratory Task Manager**

**Program Officer**

**Research Objectives**

**Technical Summary**

**Funding Summary by Cost Category (by FY, \$K)**

	Starting FY	FY+1	FY+2
Salary			
Equipment/Facilities			
Supplies			
Total			

**Report Document**

**Report Document - Text Analysis**

**Report Document - Text Analysis**

**Appendix Documents**

**2. Thank You**

**E-mail user**

Nov 24, 2015 14:21:12 Success: Email Sent to: George\_Karniadakis@brown.edu